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The scattering of short surface waves by a cylinder

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The scattering of short surface waves by a partially immersed cylinder is considered. The cylinder is taken to be circular and to pass through two fixed points on the free surface with its centre on or above the free surface. Of particular interest is the behaviour of the solution as the cross-section of the immersed part of the cylinder approaches a semicircle. The method of matched asymptotic expansions is used.

1. Introduction and formulation

The scattering of short surface waves by a partially immersed cylinder is considered, with the aim of finding a high-order asymptotic solution. The cylinder is taken to be circular and to intersect the free surface at two fixed points P and Q at an angle α measured through the fluid [see (1.1)]; thus as α varies between $\frac{1}{2}\pi$ and π, the immersed part of the cylinder changes from a semicircular cylinder to a dock. The circular geometry is chosen to obtain explicit results, and in the final section the results are generalized.

A high-order solution of this problem is of interest because of an apparent discontinuity in the transmission coefficient in the limit $\alpha \to \frac{1}{2}\pi$: in terms of the small non-dimensional wavelength parameter ϵ/a, the coefficients are $O(\epsilon/a)^2$ and $O(\epsilon/a)^4$ in the limiting and semicircular cases respectively. A similar step from $O(\epsilon/a)$ to $O(\epsilon/a)^2$ was noted by Holford (1965) in the corresponding radiation problem.

Co-ordinates are chosen with the z axis parallel to the generators of the cylinder, the y axis directed into the fluid and the x axis lying in the free surface. The cylinder S is described by

$$x^2 + (y - a\cot\alpha)^2 = a^2 \operatorname{cosec}^2\alpha, \tag{1.1}$$

and intersects the free surface at P ($x = a$) and Q ($x = -a$). It is held fixed and irradiated by the incoming wave $\operatorname{Re}\{\exp(-i(x-a)/\epsilon + i\chi - y/\epsilon - i\omega t)\}$, where ω is the angular frequency, $\epsilon = g/\omega^2$ is $\frac{1}{2}\pi$ times the wavelength of the surface wave and χ is chosen to be $\frac{1}{4}(1-\pi/2\alpha)\pi$ for algebraic convenience.

We seek a two-dimensional velocity potential of the form $\operatorname{Re}\{\phi(x,y)\exp(-i\omega t)\}$ in the short-wave limit $\epsilon \ll a$; the potential ϕ is then specified by the following conditions:

$$\phi_{xx} + \phi_{yy} = 0 \quad \text{in the fluid,} \tag{1.2}$$

$$\phi_n = 0 \quad \text{on } S, \tag{1.3}$$

$$\phi + \epsilon\phi_y = 0 \quad \text{on} \quad y = 0, \quad |x| > a, \tag{1.4}$$

where n is the outward normal from S and the suffixes denote partial derivatives. The radiation conditions

$$\phi \sim \begin{cases} \exp\{-i(x-a)/\epsilon + i\chi - y/\epsilon\} + \tilde{R}\exp\{i(x-a)/\epsilon - i\chi - y/\epsilon\} & \text{as} \quad x \to \infty, \\ \tilde{T}\exp\{-i(x-a)/\epsilon - i\chi - y/\epsilon\} \quad \text{as} \quad x \to -\infty \end{cases} \tag{1.5}$$

and the boundedness condition

$$\delta(\partial\phi/\partial\delta) \to 0 \quad \text{as} \quad \delta \to 0, \tag{1.6}$$

where δ is the distance from either P or Q, ensure uniqueness provided that $\alpha \geqslant \frac{1}{2}\pi$. Note that for $\alpha < \frac{1}{2}\pi$ the approximate solution obtained here is uniquely determined. We provide a solution for $0 < \alpha < \pi$; the case $\alpha = \pi$ has minor differences and has been considered by Leppington (1972).

The method of solution is the systematic method of matched asymptotic expansions developed by Van Dyke (1964) as applied to problems involving short surface waves by Leppington (1972, 1973*a*, *b*), Ayad & Leppington (1977) and Alker (1974, 1975). The basic idea is that the fluid region can be covered by a number of overlapping domains in each of which an asymptotic approximation of the potential $\phi(x,y)$ may be found. For a detailed application of the method the above papers should be consulted; only the main features are summarized below.

In the 'outer region', which consists of points at distances $\gg \epsilon$ from the free surface, the potential ϕ is written as an asymptotic expansion in ϵ, $\phi(x,y;\epsilon) \sim \Sigma\,\alpha_j(\epsilon)\,\phi_j(x,y)$, and substituted into the boundary conditions (1.2)–(1.5) to yield a sequence of problems for the potentials ϕ_j. Although this 'outer expansion' is expected to be valid in most of the fluid, it cannot contain surface-wave terms and so fails near the free surface. Clearly the boundedness condition at the points P and Q can no longer be imposed on the outer expansion (P and Q being outside its domain of validity) and eigensolutions, singular at P or Q, may be freely added to each term of the expansion; their coefficients are later determined by the matching principle. The outer potentials ϕ_j are thus required to satisfy

$$\nabla^2\phi_j = 0 \quad \text{in the fluid}, \tag{1.7}$$

$$\phi_{jn} = 0 \quad \text{on } S, \tag{1.8}$$

$$\phi_j = -\rho(x) \quad \text{on} \quad y = 0, \quad |x| > a, \tag{1.9}$$

$$\phi_j \to 0 \quad \text{as} \quad x^2 + y^2 \to \infty, \tag{1.10}$$

where $\rho(x)$ is either zero or the derivative of a previous term. The behaviour of the potential ϕ as P and Q are approached is important and use is made of the polar coordinates (δ,θ) and (δ_1,θ_1) defined by

$$(x-a,y) = (\delta\cos\theta, \delta\sin\theta), \quad (-x-a,y) = (\delta_1\cos\theta_1, \delta_1\sin\theta_1). \tag{1.11}$$

In the vicinity of the two intersection points P and Q, the solution will be sensitive to the wave-bearing nature of the free surface, but will depend primarily on the local geometry of S. This suggests that in these 'inner regions', comprising points at distances $\ll a$ from P and Q, the solution varies on a wavelength scale, and for a detailed examination we take co-ordinates

$$\left.\begin{aligned}(X, Y) &= (R\cos\theta, R\sin\theta) = ((x-a)/\epsilon, y/\epsilon),\\ (X_1, Y_1) &= (R_1\cos\theta_1, R_1\sin\theta_1) = (-(x+a)/\epsilon, y/\epsilon),\end{aligned}\right\} \tag{1.12}$$

and define the inner potentials Φ and Ψ by

$$\Phi(X, Y; \epsilon) = \phi(x, y; \epsilon), \quad \Psi(X_1, Y_1; \epsilon) = \phi(x, y; \epsilon). \tag{1.13}$$

In these inner regions, the deviation of S from its tangent at the free surface is small, and the boundary condition (1.3) is replaced by a new condition on the tangent by expanding S and the inner expansion in a Taylor series about the tangent.

Thus the inner potentials are defined in terms of sloping-beach problems, and as there is no boundedness condition as R and R_1 increase, eigensolutions unbounded at infinity must be added to the expansions. The coefficients of the eigensolutions added to the inner and outer expansions are determined when the expansions are matched; thus the matching principle supplies the missing boundary conditions in each region.

The inner potential Φ is required to satisfy

$$\nabla^2\Phi = 0 \quad \text{for} \quad 0 < \theta < \alpha, \tag{1.14}$$

$$\Phi + \Phi_Y = 0 \quad \text{on} \quad Y = 0, \quad X > 0, \tag{1.15}$$

$$\Phi_\xi + \frac{\epsilon\sin\alpha}{2a}(2_\eta\Phi_\eta - \eta^2\Phi_{\xi\xi}) + \ldots = 0 \quad \text{on} \quad \xi = 0, \quad \eta > 0, \tag{1.16}$$

$$R\Phi_R \to 0 \quad \text{as} \quad R \to 0, \tag{1.17}$$

$$\Phi - \exp\{-i(X-\chi) - Y\} \sim \tilde{R}\exp\{i(X-\chi) - Y\} + \text{wave-free terms as } X \to \infty, \tag{1.18}$$

where the co-ordinates (ξ, η) correspond to a rotation of the axes (to direct the η axis along $\theta = \alpha$) defined by

$$(\xi, \eta) = (X\sin\alpha - Y\cos\alpha, X\cos\alpha + Y\sin\alpha). \tag{1.19}$$

Similar conditions hold for the potential Ψ, except that the radiation condition is

$$\Psi \sim \tilde{T}\exp\{i(X_1-\chi) - Y_1\} + \text{wave-free terms as } X_1 \to \infty. \tag{1.20}$$

The matching principle to be used is a modified version of that proposed by Van Dyke (1964): the modification due to Crighton & Leppington (1973) stipulates that all terms of the form $\epsilon^\gamma\log\epsilon$ or $\epsilon^\gamma\log\log\epsilon$ must be grouped with ϵ^γ for matching purposes. We first define the expansion of the inner potential $\Phi(R, \theta; \epsilon)$ up to and including terms of order ϵ^s as $\Phi^{(s)}(R, \theta; \epsilon)$. Then in order to match the inner potential $\Phi(R, \theta; \epsilon)$ with the outer potential $\phi(\delta, \theta; \epsilon)$, we take the limit of $\Phi^{(s)}$ as $R \to \infty$ and replace R by δ/ϵ. This expression is then expanded in ϵ (for fixed δ) and truncated to include terms of order up to and including ϵ^t, and the resulting series is denoted by $\Phi^{(s,t)}$. Similarly, by replacing δ by ϵR in $\Phi^{(t)}$, expanding and truncating after ϵ^s, we obtain $\phi^{(t,s)}$. The matching condition is

$$\Phi^{(s,t)} \equiv \phi^{(t,s)} \tag{1.21}$$

for any s and t of our choice.

Finally, the outer expansion is extended up to the free surface (for points at distances $\gg \epsilon$ from P and Q) by simply continuing the surface waves – initially valid only in the inner regions – over the whole free surface.

2. The basic solution

Inner eigensolutions

The eigensolutions in each region play an important role in the solution of the problem. In the inner regions, the eigensolutions Φ_{eq} which satisfy the conditions (1.14), (1.15), (1.17) and

$$\partial\Phi_{eq}/\partial\xi = 0 \quad \text{on} \quad \xi = 0, \quad \eta > 0 \tag{2.1}$$

may be extracted from Peters' (1950) work on standing-wave solutions and are defined by

$$\Phi_{e0} = \mathrm{Re}\left\{\frac{\mu^{\frac{1}{2}}}{\pi i}\int_P \frac{f_1(\zeta)}{\zeta}e^{Z\zeta}d\zeta\right\}$$
$$\sim 2\cos(X-\chi)\,e^{-Y} + 2\mu^{\frac{1}{2}}\Gamma(\mu)\sin\pi\mu\sin\mu\theta/\pi R^{\mu} + \ldots \quad \text{as} \quad R\to\infty, \tag{2.2}$$

$$\Phi_{eq} = \mathrm{Re}\left\{\frac{\mu^{\frac{1}{2}}}{\pi i}\int_P \frac{f_1(\zeta)\,e^{Z\zeta}}{\zeta^{1+2\mu q}}d\zeta\right\} - \exp(i\pi\mu q)\,\Phi_{e0}, \quad q = 1, 2, \ldots,$$
$$\sim -2i\sin\pi\mu q\exp\{i(X-\chi)-Y\} - 2\mu^{\frac{1}{2}}R^{(2q-1)\mu}\sin(2q-1)\mu\theta/\Gamma(1+(2q-1)\mu) + \ldots, \tag{2.3}$$

where

$$\mu = \pi/2\alpha, \quad \chi = \tfrac{1}{4}(1-\mu)\pi, \quad Z = X+iY$$

and

$$f_1(\zeta) = \frac{\zeta}{\zeta-i}\exp\left\{-\frac{1}{\pi}\int_0^\infty \log\left[\frac{1-t^{-2\mu}}{1-t^{-2}}\right]\frac{\zeta}{t^2+\zeta^2}dt\right\}. \tag{2.4}$$

The path P is defined as follows: if $\zeta^{-2\mu q}f_1(\zeta)$ has no branch point at the origin, then P is taken anticlockwise on a circle of radius > 1 about the origin. Otherwise a cut is taken along $\arg\zeta = \pm\pi - \frac{1}{2}\alpha$ and a linear path extending to infinity on each side of the branch cut is added to the circle. It is noteworthy that when μ is an integer, and in particular in the case of a vertical beach, all eigensolutions with no incoming waves are totally wave free.

Outer eigensolutions

The eigensolutions ϕ_{en} in the outer region which satisfy (1.7), (1.8), (1.10) and

$$\phi_{en} = 0 \quad \text{on} \quad |x| < a, \quad y = 0 \tag{2.5}$$

are defined by

$$\phi_{en} = \mathrm{Re}\left\{i\left(\frac{z+a}{z-a}\right)^{(2n+1)\mu}\right\}, \quad n = 0, \pm 1, \ldots, \tag{2.6}$$

where $z = x+iy$.

First-order solution

It is straightforward to find the first-order term in each of the three expansions (see Leppington 1972): in the illuminated inner region we find

$$\Phi \sim \Phi^{(0)} = \Phi_0 = \Phi_{e0}(X, Y), \tag{2.7}$$

in the outer region

$$\phi \sim \phi^{(\mu)} = \epsilon^{\mu}C\phi_{e0}(x, y) \tag{2.8}$$

and in shadow inner region

$$\Psi \sim \Psi^{(2\mu)} = \epsilon^{2\mu} D \Phi_{e1}(X_1, Y_1). \tag{2.9}$$

The matching rules $\Phi^{(0,\mu)} \equiv \phi^{(\mu,0)}$ and $\Psi^{(2\mu,\mu)} \equiv \phi^{(\mu,2\mu)}$ were used to determine the coefficients

$$C = \frac{2\mu^{\frac{1}{2}}\Gamma(\mu)\sin\pi\mu}{\pi(2a)^{\mu}}, \quad D = \frac{-\mu(\Gamma(\mu))^2\sin\pi\mu}{\pi(2a)^{2\mu}} \tag{2.10}$$

and to show that to the order considered all other eigensolution coefficients are zero. As expected, these first-order potentials are the least singular solutions for the three regions.

To obtain a higher-order solution, further terms are postulated for each expansion and substituted into the boundary conditions. Care must be taken not to exclude unexpected terms.

Illuminated inner expansion

For the inner potential Φ, we pose

$$\Phi^{(1)} = \Phi_0 + h(\epsilon)\,\Phi_h + \epsilon\Phi_1, \tag{2.11}$$

where the term $h(\epsilon)\,\Phi_h$ is added to indicate the possibility of terms with scalings other than ϵ. Both Φ_h and Φ_1 have no incoming waves and must satisfy (1.14), (1.15), (1.17) and on $\xi = 0$

$$\Phi_{h\xi} = 0, \quad \Phi_{1\xi} = (-\sin(\alpha)/2a)\,(2\eta\Phi_{0\eta} - \eta^2\Phi_{0\xi\xi}). \tag{2.12}$$

The potential Φ_h is a sum of eigensolutions with coefficients a_q:

$$\Phi_h = \sum_{q\geqslant 1} a_q \Phi_{eq}. \tag{2.13}$$

The potential Φ_1 is found in §4; in particular, its far-field expansion is found to have a term $R^{-\mu}\log R\sin\mu\theta$.

Outer expansion

The far-field expansion of $\Phi^{(1)}$ suggests that the outer expansion has the form

$$\phi^{(1+\mu)} = \epsilon^{\mu}\Phi_0 + \epsilon^{1+\mu}\log\epsilon\,\phi_1 + l(\epsilon)\,\phi_l + \epsilon^{1+\mu}\phi_2. \tag{2.14}$$

Each potential satisfies (1.7), (1.8) and (1.10); ϕ_1 and ϕ_l are clearly eigensolutions thus

$$\phi_1 = \Sigma\, c_n \phi_{en}, \quad \phi_l = \Sigma\, d_n \phi_{en}. \tag{2.15}$$

The potential ϕ_2 satisfies $\phi_2 = -\phi_{0y}$ on $|x| > a$, $y = 0$, and in terms of

$$w = (z+a)/(z-a)$$

we find

$$\phi_2 = (\mu C/2a)\,\phi_{21} + \Sigma f_n \phi_{en}, \tag{2.16}$$

where

$$\phi_{21} = \operatorname{Re}\left\{\frac{ie^{i\alpha}}{\sin\alpha}\,w^{1+\mu} + 2w^{\mu} - \frac{4i\mu}{\pi}w^{\mu}\log w - \frac{ie^{-i\alpha}}{\sin\alpha}\,w^{\mu-1}\right\}. \tag{2.17}$$

Shadow inner expansion

Similarly, for the potential Ψ we pose

$$\Psi^{(1+2\mu)} = \epsilon^{2\mu}\Psi_0 + \epsilon^{1+2\mu}\log\epsilon\,\Psi_1 + j(\epsilon)\,\Psi_j + \epsilon^{1+2\mu}\Psi_2. \tag{2.18}$$

The potentials Ψ_1 and Ψ_j are eigensolutions with no incoming waves:

$$\Psi_1 = \sum_{q\geqslant 1} g_q \Phi_{eq}, \quad \Psi_j = \sum_{q\geqslant 1} j_q \Phi_{eq}. \tag{2.19}$$

The potential Ψ_2 is determined to within eigensolutions by

$$\Psi_{2\sigma} = (-\sin(\alpha)/2a)(2\tau\Psi_{0\tau} - \tau^2\Psi_{0\sigma\sigma}) \quad \text{on} \quad \sigma = 0, \quad \tau > 0, \tag{2.20}$$

where (σ, τ) correspond to (ξ, η).

Once the two inner potentials Φ_1 and Ψ_2 are known, the expansions may be matched and the eigensolution coefficients found.

3. Inhomogeneous sloping-beach problems

The first work on the inhomogeneous problem was done by Shen (1965). He gave explicit results for two particular cases of oscillating point sources on the free surface: the transient problem with a beach angle $\frac{1}{4}\pi$ and the steady-state problem with a general beach angle. His formulation, using Peters' (1950) approach, was valid for any surface forcing. Morris (1974) generalized the steady-state problem to arbitrary positioning of the source in the sector. Using Peters' work in a similar way, the solution of the beach forcing problem is now given.

Define the real, harmonic potential $\Phi(X, Y)$ and the analytic function W by the following conditions:

$$\left.\begin{aligned} &\Phi + \Phi_Y = 0 \quad \text{on} \quad Y = 0, \quad X > 0, \\ &\Phi_\xi = \operatorname{Re}(H(Z)) \quad \text{on} \quad \xi = 0, \quad \eta > 0, \\ &\Phi = \operatorname{Re}(W(Z)), \end{aligned}\right\} \tag{3.1}$$

where $Z = X + iY$. Then in terms of the function W these conditions are

$$\left.\begin{aligned} &\operatorname{Re}(W + iW') = 0 \quad \text{on} \quad \arg Z = 0, \\ &\operatorname{Re}(-ie^{i\alpha}W' - H) = 0 \quad \text{on} \quad \arg Z = \alpha. \end{aligned}\right\} \tag{3.2}$$

The functions in the parentheses may be analytically continued across $\arg Z = 0$ and $\arg Z = \alpha$ by Schwarz's reflexion principle, to define W in the sector $-\alpha \leqslant \arg Z \leqslant 3\alpha$. For $0 \leqslant \arg Z \leqslant \alpha$ we have

$$\left.\begin{aligned} \overline{W}(Z) - i\overline{W}'(Z) &= -W(Z) - iW'(Z) \\ \text{and} \qquad -ie^{i\alpha}W'(Ze^{2i\alpha}) - H(Ze^{2i\alpha}) &= -i\bar{e}^{i\alpha}\overline{W}'(Z) + \overline{H}(Z), \end{aligned}\right\} \tag{3.3}$$

where $\overline{W}(Z)$ is intepreted as $\overline{W(\overline{Z})}$. Eliminating $\overline{W}$ and $\overline{W}'$ yields

$$\begin{aligned} W''(Z) - iW'(Z) = {} & e^{4i\alpha}W''(Ze^{2i\alpha}) + ie^{2i\alpha}W'(Ze^{2i\alpha}) \\ & + e^{i\alpha}[H(Ze^{2i\alpha}) + \overline{H}(Z) - ie^{2i\alpha}H'(Ze^{2i\alpha}) - i\overline{H}'(Z)]. \end{aligned} \tag{3.4}$$

Define the operator L by

$$Lg(\zeta) = \frac{1}{2\pi i}\int_P e^{Z\zeta}\frac{g(\zeta)}{\zeta}\,d\zeta. \tag{3.5}$$

Then the functions $f(\zeta)$ and $h(\zeta)$ are implicitly defined by

$$W(Z) = Lf(\zeta), \quad H(Z) = Lh(\zeta), \tag{3.6}$$

for $0 \leqslant \arg Z \leqslant \alpha$, where P is the same contour as that defining the eigensolutions Φ_{eq}. Representations for $W(Ze^{2i\alpha})$, $H(Ze^{2i\alpha})$ and $\bar{H}(Z)$ may be found to be

$$\left.\begin{aligned} W(Ze^{2i\alpha}) = Lf(\zeta e^{-2i\alpha}), \quad H(Ze^{2i\alpha}) = Lh(\zeta e^{-2i\alpha}),\\ \bar{H}(Z) = L\bar{h}(\zeta), \end{aligned}\right\} \tag{3.7}$$

for $0 \leqslant \arg Z \leqslant \alpha$. On substitution, we find that (3.4) is satisfied when

$$(\zeta - i) f(\zeta) = (\zeta + i) f(\zeta e^{-2i\alpha}) + G(\zeta) \tag{3.8}$$

for $-\pi - \frac{1}{2}\alpha \leqslant \arg \zeta \leqslant \pi - \frac{1}{2}\alpha$ (the range ζ covers on P), where

$$G(\zeta) = -i e^{i\alpha}(\zeta + i)\,\zeta^{-1}(h(\zeta e^{-2i\alpha}) + \bar{h}(\zeta)). \tag{3.9}$$

This equation is simplified by writing

$$f(\zeta) = f_1(\zeta) f_2(\zeta); \tag{3.10}$$

note that the function $f_1(\zeta)$ defined by (2.4) satisfies the homogeneous equation

$$(\zeta - i) f_1(\zeta) = (\zeta + i) f_1(\zeta e^{-2i\alpha}). \tag{3.11}$$

The equation for $f_2(\zeta)$ is therefore

$$f_2(\zeta) = f_2(\zeta e^{-2i\alpha}) + G(\zeta)/(\zeta - i) f_1(\zeta). \tag{3.12}$$

For a suitably integrable function G ($G = o(\zeta)$ as $\zeta \to \infty$; $G = o(\zeta^{-\mu})$ as $\zeta \to 0$), Plemelj's formula may be used to give a representation for f_2 in $-2\alpha \leqslant \arg \zeta \leqslant 0$:

$$f_2(\zeta) = \frac{-1}{2\pi i} \int_0^\infty \frac{G(t^{\frac{1}{2}\mu})\, dt}{(t^{\frac{1}{2}\mu} - i) f_1(t^{\frac{1}{2}\mu})(t - \zeta^{2\mu})}. \tag{3.13}$$

Given the analytical continuation of the function f_2, we have an integral representation for Φ:

$$\Phi = \operatorname{Re}\{Lf(\zeta)\}.$$

If $f_2(\zeta)$ has poles only where $\operatorname{Re}\zeta < 0$, then the far-field behaviour may be shown to be

$$\Phi \sim \operatorname{Re}\{\mu^{-\frac{1}{2}} e^{i\chi} e^{iZ} f_2(i)\} + \text{wave-free terms.}$$

To satisfy an outgoing radiation condition, a multiple of the standing-wave eigensolution Φ_{e0} may be added.

4. The evaluation of Φ_1 and Ψ_2

The potential Φ_1 is specified by the beach pressure

$$\Phi_{1\xi} = (-\sin(\alpha)/2a)(2\eta\Phi_{0\eta} - \eta^2\Phi_{0\xi\xi}) \quad \text{on} \quad \xi = 0, \quad \eta > 0.$$

As Φ_0 is real, the inhomogeneous part of Φ_1 is also real and hence analytic functions ω and ω_0 may be defined by

$$\Phi_1 = \operatorname{Re} W(Z), \quad \Phi_0 = \Phi_{e0} = \operatorname{Re} W_0(Z). \tag{4.1}$$

Complex eigensolutions will be added to Φ_1 later. Following §3,

$$W(Z) = Lf(\zeta),$$

and using the fact that $W_0(Z) = 2\mu^{\frac{1}{2}} L f_1(\zeta)$ from (2.2), we find

$$h(\zeta) = -\mu^{\frac{1}{2}} \sin\alpha\, \zeta^2 f_1''(\zeta)/a. \tag{4.2}$$

Using the difference equation (3.11) and the fact that $(\zeta - i) f_1(\zeta)$ is real for real ζ, we find that

$$G(\zeta) = \frac{2i}{a} e^{i\alpha} \mu^{\frac{1}{2}} \sin\alpha\, \zeta(\zeta+i) \frac{d^2}{d\zeta^2}\left[\frac{\zeta-i}{\zeta+i} f_1(\zeta)\right] \tag{4.3}$$

$$= \begin{cases} O(\zeta^{-1}) & \text{as} \quad \zeta \to \infty, \\ O(\zeta^{\mu-1}) & \text{as} \quad \zeta \to 0. \end{cases}$$

Plemelj's formula may be used for $\mu > \frac{1}{2}$ (the case $\mu = \frac{1}{2}$ is easily handled, but has a zero $\sin\alpha$ coefficient) to obtain the integral representation given by (3.13), and thus the inhomogeneous part of Φ_1 is determined.

In the appendix an analytic continuation of the function $f_2(\zeta)$ is found and is shown to satisfy (3.12) in the range $-\pi - \frac{1}{2}\alpha \leqslant \arg\zeta \leqslant \pi - \frac{1}{2}\alpha$. The far-field behaviour of the potential is then found to be

$$\begin{aligned} \Phi_1(X, Y) \sim \mu^{-\frac{1}{2}} |f_2(i)| \cos(X - \chi + \chi_2)\, e^{-Y} &- (\mu^{\frac{3}{2}}/a\pi)\, \Gamma(\mu) \sin\pi\mu\, R^{1-\mu} \sin(1-\mu)\,\theta \\ &+ (4\mu^{\frac{5}{2}}/a\pi)\, \Gamma(\mu) \sin\pi\mu\, R^{-\mu}[\log R \sin\mu\theta - \theta\cos\mu\theta] \\ &+ (\mu^{\frac{3}{2}}/a\pi)\, \Gamma(\mu) \sin\pi\mu K_1 R^{-\mu} \cos\mu\theta \\ &- (\mu^{\frac{3}{2}}/a\pi^2)\, \Gamma(\mu)\, (K_2 \sin\pi\mu + 4\pi\mu\cos\pi\mu)\, R^{-\mu} \sin\mu\theta + \ldots + \sum_{q\geqslant 0} b_q \Phi_{eq}, \end{aligned} \tag{4.4}$$

where $|f_2(i)| = |f_2(e^{\frac{1}{2}i\pi})|$ and $\chi_2 = \arg f_2(e^{\frac{1}{2}i\pi})$. To ensure outgoing waves only, we choose

$$b_0 = -\tfrac{1}{2}\mu^{-\frac{1}{2}} |f_2(i)|\, e^{-i\chi_2}. \tag{4.5}$$

Φ_1 then has the form

$$\Phi_1 \sim i\mu^{-\frac{1}{2}} |f_2(i)| \sin\chi_2\, e^{i(X-\chi)-Y}, \tag{4.6}$$

and the coefficient of the $R^{-\mu}\sin\mu\theta$ term becomes

$$K_3 = -\cos\mu\pi 4\mu^{\frac{5}{2}} \Gamma(\mu)/a\pi - \sin\mu\pi\, \Gamma(\mu)\, [(\mu^{\frac{3}{2}} K_2 + a\pi\, |f_2(i)|\, e^{-i\chi_2}]/a\pi^2. \tag{4.7}$$

[Note that when $\mu = 1$ the functions f_1 and f_2 are easy to find explicitly, and the far-field expansion $\Phi_1 \sim (-i/2a)\, e^{iX-Y} + 4\sin\theta/a\pi R - 4\cos 2\theta/a\pi R^2 + \ldots$ agrees with that found using the Green's function in Alker (1974), i.e. $a = -1/\alpha_2$.]

The potential Ψ_2 is similar to the potential Φ_1, for using the co-ordinates (X, Y), the beach forcing is

$$\Psi_{2\xi} = (-\sin(\alpha)/2a)\, (2\eta\Psi_{0\eta} - \eta^2\Psi_{0\xi\xi}) \quad \text{on} \quad \xi = 0, \quad \eta > 0.$$

Now

$$\Psi_0 = D\Psi_{e1} = D\,\mathrm{Re}\, W_1(Z) - D e^{i\pi\mu}\Phi_{e0},$$

where $W_1(Z) = 2\mu^{\frac{1}{2}} L(\zeta^{-2\mu} f_1(\zeta))$ from (2.3). The analytic function $D\Omega(Z)$ is defined to be the inhomogeneous part of Ψ_2 generated by DW_1. Thus the potential Ψ_2 may be written as

$$\Psi_2 = D\Psi_{21} + \sum_{q\geqslant 1} k_q \Phi_{eq}, \tag{4.8}$$

where

$$\Psi_{21} = \mathrm{Re}\,\Omega - e^{i\mu\pi}\Phi_1 + k_0\Phi_{e0}. \tag{4.9}$$

The function Ω satisfies the same conditions as W, with W_0 replaced by W_1. In this case

$$G^*(\zeta) = \frac{2i}{a} e^{i\alpha}\mu^{\frac{1}{2}}\sin\alpha\zeta(\zeta+i)\frac{d^2}{d\zeta^2}\left[\frac{\zeta-i}{\zeta+i}\zeta^{-2\mu}f_1(\zeta)\right]$$
$$= O(\zeta^{-1-\mu}) \quad \text{as} \quad \zeta \to 0. \tag{4.10}$$

The previous device of writing $f^* = f_1 f_3$ will yield a problem for f_3 which cannot be solved using Plemelj's formula. However, by putting

$$\left.\begin{aligned} f^*(\zeta) &= f_1(\zeta)\,\zeta^{-2\mu}f_3(\zeta),\\ f_3(\zeta) &= f_3(\zeta e^{-2i\alpha}) + \zeta^{2\mu}G^*(\zeta)/(\zeta-i)f_1(\zeta),\end{aligned}\right\} \tag{4.11}$$

a Plemelj representation for f_3 may be found directly. (In the general problem, the function f may be found by a substitution of the form

$$f = f_1(\zeta)\,[A\zeta^{\alpha} + B\zeta^{\beta}\log\zeta + \ldots + \zeta^{2\mu p}g_p(\zeta)].)$$

The far-field behaviour of Ω is given in the appendix; in particular

$$\operatorname{Re}\Omega \sim \mu^{-\frac{1}{2}}\cos(X-\chi-\mu\pi+\chi_3)\,|f_3(i)|\,e^{-Y} + \text{wave-free terms as } X \to \infty,$$

where $f_3(e^{\frac{1}{2}i\pi}) = |f_3(i)|\,e^{i\chi_3}$, and to ensure outgoing waves we choose

$$k_0 = -\tfrac{1}{2}\mu^{-\frac{1}{2}}\,|f_3(i)|\exp(i\mu\pi - i\chi_3). \tag{4.12}$$

The wave-free far-field behaviour of $\Psi_2(X_1, Y_1)$ is thus

$$\Psi_2(X_1,Y_1) \sim \frac{\mu^{\frac{1}{2}}D}{a\pi\Gamma(\mu)}[\pi R_1^{\mu+1}\sin(\mu+1)\,\theta_1 + 4\mu R_1^{\mu}(\log R_1\sin\mu\theta_1 + \theta_1\cos\mu\theta_1)$$
$$+ L_1R_1^{\mu}\cos\mu\theta_1 + L_2R_1^{\mu}\sin\mu\theta_1] - \frac{2k_1}{\mu^{\frac{1}{2}}\Gamma(\mu)}R_1^{\mu}\sin\mu\theta_1 + \ldots \tag{4.13}$$

[when $\mu = 1, f_3 = -f_2 + 4\log\zeta/\pi\zeta^2 - 2/\zeta^3$ and

$$a\operatorname{Re}\Omega \sim R_1^2\sin 2\theta_1 - 2R_1\cos\theta_1 + 4R_1(\log R_1\sin\theta_1 + \theta_1\cos\theta_1)/\pi - 4\log R_1/\pi - 4\gamma/\pi$$
$$- 4i\exp(iX_1 - Y_1) + \ldots + \text{wave-free eigensolutions},$$

which agrees with the form found using the Green's function; see Alker 1975].

5. The second-order solution

The expansions have been found to second order, to within eigensolutions, and the imposition of the matching principles

$$\Phi^{(1,1+\mu)} \equiv \phi^{(1+\mu,1)}, \quad \Psi^{(1+2\mu,1+\mu)} \equiv \phi^{(1+\mu,1+2\mu)} \tag{5.1}$$

determines all the coefficients of the eigensolutions. We find that

$$\left.\begin{aligned} c_0 &= -C2\mu^2/a\pi,\\ f_0 &= -\frac{K_3}{(2a)^{\mu}} - \frac{\mu^{\frac{3}{2}}\sin\mu\pi\Gamma(\mu)}{(2a)^{\mu}a\pi^2}[\pi(1+\mu) - 4\mu\log 2a],\\ g_1 &= -8\mu^2D/\pi,\\ k_1 &= \frac{-\mu^{\frac{1}{2}}\Gamma(\mu)f_0}{2(2a)^{\mu}} + \frac{C\mu}{(2a)^{1+\mu}}[\pi(1-\mu)\cot\alpha - L_2]\end{aligned}\right\} \tag{5.2}$$

and that all the other coefficients are zero except for b_0 and k_0, which are already defined by (4.5) and (4.12).

It is useful to summarize the results as follows: for the illuminated inner expansion we have

$$\Phi^{(1)} = \Phi_{e0} + \epsilon\Phi_1, \tag{5.3}$$

for the outer expansion

$$\phi^{(1+\mu)} = C\left\{\epsilon^{\mu}\phi_{e0} - \epsilon^{1+\mu}\log\epsilon\,\frac{2\mu^2}{a\pi}\,\phi_{e0} - \epsilon^{1+\mu}\,\frac{\mu}{2a}\,\phi_{21}\right\} + \epsilon^{1+\mu}f_0\phi_{e0}, \tag{5.4}$$

and for the shadow inner expansion

$$\Psi^{(1+2\mu)} = D\{\epsilon^{2\mu}\Phi_{e1} - \epsilon^{1+2\mu}\log\epsilon 8\mu^2\Phi_{e1}/\pi + \epsilon^{1+2\mu}\Psi_{21}\} + k_1\Phi_{e1}. \tag{5.5}$$

The reflexion coefficient is given by

$$\tilde{R} = 1 + i\epsilon\mu^{-\frac{1}{2}}\,|f_2(i)|\sin\chi_2 + o(\epsilon) \tag{5.6}$$

and the transmission coefficient by

$$\tilde{T} = iD\{-2\sin\mu\pi\epsilon^{2\mu}(1 + 8\mu^2\epsilon\log\epsilon/\pi) + \epsilon^{1+2\mu}\mu^{-\frac{1}{2}}[|f_3(i)|\sin(\chi_3 - \mu\pi) - e^{i\mu\pi}\,|f_2(i)|\sin\chi_2]\}$$
$$- i\epsilon^{1+2\mu}2k_1\sin\pi\mu + o(\epsilon^{1+2\mu}). \tag{5.7}$$

Recall that

$$C = 2\mu^{\frac{1}{2}}\Gamma(\mu)\sin\pi\mu/\pi(2a)^{\mu}, \quad D = -\mu(\Gamma(\mu))^2\sin\pi\mu/\pi(2a)^{2\mu}.$$

6. The semicircular limit

In the introduction, reference was made to an apparent discontinuity in the transmission coefficient as $\alpha \to \frac{1}{2}\pi\,(\mu \to 1)$. From the solution found, this coefficient is of order $\epsilon^{2\mu}$ for $\mu \neq 1$ and $o(\epsilon^{1+2\mu})$ when $\mu = 1$. That this does not really represent a non-uniformity in the solution may be seen from the occurrence of the coefficients C and D, both of which contain the factor $\sin\pi\mu$ and approach zero smoothly as $\mu \to 1$. Thus rather than 'disappearing', a number of terms are seen to have coefficients which have zeros at $\mu = 1$. The first non-zero term, for $\mu = 1$, in the shadow inner region is

$$\Psi^{(3)} = \epsilon^3(R_1\sin\theta_1 - 1)/a^3\pi, \tag{6.1}$$

which is wave free.

To find the wave amplitude for $\mu = 1$, we return to the solution for a general angle and continue the expansion, which must have the following form:

$$\Psi \sim \Psi^{(2+2\mu)} = \epsilon^{2\mu}\Psi_0 + \epsilon^{1+2\mu}\log\epsilon\Psi_1 + \epsilon^{1+2\mu}\Psi_2 + \epsilon^{3\mu}\Psi_3 + \epsilon^{4\mu}\Psi_4$$
$$+ \epsilon^{2+2\mu}(\log\epsilon)^2\Psi_5 + \epsilon^{2+2\mu}\log\epsilon\Psi_6 + t(\epsilon)\Psi_e + \epsilon^{2+2\mu}\Psi_7. \tag{6.2}$$

Clearly the potentials Ψ_3, Ψ_4, Ψ_5 and Ψ_e are eigensolutions and have zero wave amplitude for $\mu = 1$. The inhomogeneous part of Ψ_6 will have a coefficient D from Ψ_1, and hence will not contribute. The potential Ψ_7 may be considered as the sum of a multiple of Ψ_{21} generated by the eigensolution $k_1\Phi_{e1}$ of Ψ_2, a second-degree inhomogeneous potential with a coefficient D from the remainder of Ψ_2, and eigensolutions.

Hence at $\mu = 1$ the only term to order ϵ^4 to contribute to the wave amplitude is the Ψ_{21} term of Ψ_7. Thus

$$\Psi \sim \text{wave-free terms} - \frac{\epsilon^4}{2a^4\pi}\Psi_{21} \sim \frac{2i\epsilon^4}{\pi a^4}\exp(iX_1 - Y) + \ldots, \tag{6.3}$$

and the transmitted wave past a semicircular cylinder is

$$\phi \sim \left(\frac{\epsilon}{a}\right)^4 \frac{2i}{\pi}\exp\left\{-i\left(\frac{x+a}{\epsilon}\right) - \frac{y}{\epsilon}\right\}, \tag{6.4}$$

which is in agreement with the result proved rigorously by Ursell (1961).

7. Generalization

The generalization to non-circular cylinders is straightforward, and the order of the transmitted wave is easily predicted. If at P the beach constant is μ and $u \sim \beta_N v^N$ (where $(u, v) = (\epsilon\xi, \epsilon\eta)$ are scaled on the cylinder length scale), and at Q the beach constant is ν and $s \sim \lambda_M t^M$ $((s,t) = (\epsilon\sigma, \epsilon\tau))$, then the transmitted wave amplitude is

$$\tilde{T} \sim \text{constant} \times P(\epsilon)\, Q(\epsilon) + \ldots, \tag{7.1}$$

where $P(\epsilon)$ is the larger of $\epsilon^{\mu}\sin\pi\mu$ and $\epsilon^{N+\mu-1}$ and $Q(\epsilon)$ is the larger of $\epsilon^{\nu}\sin\pi\nu$ and $\epsilon^{M+\nu-1}$. The constant, which is independent of ϵ, depends on the geometry of the cylinder, and its determination requires details of the first-order outer potential, and possibly of the inhomogeneous potentials corresponding to Φ_1 and Ψ_2. Surprisingly, the difficulty lies in the determination of the outer potential, the inner potentials being very similar to those found here.

Radiation problems may be treated in a similar way.

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Appendix

The solution of the inhomogeneous sloping-beach problems was reduced to solving the difference equations (3.12) and (4.11) for $f_2(\zeta)$ and $f_3(\zeta)$. Owing to their similarity, attention is here focused on f_2; the following also holds true for f_3.

We require a solution $f_2(\zeta)$ of the equation

$$f_2(\zeta) = f_2(\zeta e^{-2i\alpha}) + G(\zeta)/(\zeta - i) f_1(\zeta), \tag{A 1}$$

where

$$G(\zeta) = 2i\, e^{i\alpha}\frac{\mu^{\frac{1}{2}}}{a}\sin\alpha\zeta(\zeta+i)\frac{d^2}{d\zeta^2}\left[\frac{\zeta-i}{\zeta+i}f_1(\zeta)\right]$$

for $-\pi - \frac{1}{2}\alpha \leqslant \arg\zeta \leqslant \pi - \frac{1}{2}\alpha$. The function $f_1(\zeta)$, defined in (3.30), has simple poles at

$$\zeta = \exp(\tfrac{1}{2}i\pi) \quad \text{and} \quad \exp\{\pm i(\tfrac{1}{2}\pi + (2r+1)\alpha + 2n\pi)\}$$

and simple zeros at

$$\zeta = \exp\{\pm i(\tfrac{3}{2}\pi + 2r\alpha + 2n\pi)\}$$

for $r, n \geqslant 0$; see Holford (1965) and Peters (1950).

Let

$$M(\zeta) = G(\zeta)/(\zeta - i)f_1(\zeta);$$

then $M(\zeta)$ has poles at $\zeta = e^{-\frac{1}{2}i\pi}$, $e^{\pm\frac{3}{2}i\pi}$ and $e^{\pm i(\frac{1}{2}\pi+2\alpha)}$. We claim that Plemelj's formula gives the following integral representation for $f_2(\zeta)$ for $-2\alpha < \arg\zeta < 0$:

$$f_2(\zeta) = -\frac{1}{2\pi i}\int_0^\infty \frac{M(t^{1/2\mu})}{t-\zeta^{2\mu}}\,dt.$$

In order to check that f_2 satisfies (A 1) for $-\pi - \frac{1}{2}\alpha \leqslant \arg\zeta \leqslant \pi - \frac{1}{2}\alpha$ it will be necessary to construct an analytic continuation for $-\pi - \frac{5}{2}\alpha < \arg\zeta < \pi - \frac{1}{2}\alpha$.

Analytic continuation for $-2\alpha - \frac{1}{2}\pi < \arg\zeta < \alpha + \frac{1}{2}\pi$

Let

$$N_\theta(\zeta) = -\frac{1}{2\pi i}\int_{L_\theta} \frac{M(t^{1/2\mu})}{t-\zeta^{2\mu}}\,dt,$$

where L_θ is the straight path from the origin to infinity along $e^{2i\mu\theta}$, $-\frac{1}{2}\pi < \theta < \alpha + \frac{1}{2}\pi$. Then N_θ is an analytic function of ζ for $\theta - 2\alpha < \arg\zeta < \theta$. Let $\max(-2\alpha, \theta - 2\alpha) = \theta_1$ and $\min(0, \theta) = \theta_2$. Then for $\theta_1 < \arg\zeta < \theta_2$, we have $N_\theta(\zeta) = N_0(\zeta) = f_2(\zeta)$, for no poles have been passed over by the shift in contour and M is suitably bounded. Hence the integrals N_θ provide an analytic continuation for f_2 into $-2\alpha - \frac{1}{2}\pi < \arg\zeta < \alpha + \frac{1}{2}\pi$.

Equation (A 1) *is satisfied by* $f_2(\zeta)$ *for* $-\frac{1}{2}\pi < \arg\zeta < \alpha + \frac{1}{2}\pi$

Choose ζ in the above range and θ_1 and θ_2 such that $-\frac{1}{2}\pi < \theta_1 < \arg\zeta < \theta_2 < \alpha + \frac{1}{2}\pi$. Then

$$\begin{aligned} f_2(\zeta e^{-2i\alpha}) = N_{\theta_1}(\zeta) &= -\frac{1}{2\pi i}\int_{L_{\theta_1}} \frac{M(t^{1/2\mu})}{(t-\zeta^{2\mu}e^{-2i\pi})}\,dt \\ &= -\frac{1}{2\pi i}\int_{L_{\theta_1}} \frac{M(t^{1/2\mu})}{t-\zeta^{2\mu}}\,dt \end{aligned}$$

and

$$f_2(\zeta) = N_{\theta_2}(\zeta) = -\frac{1}{2\pi i}\int_{L_{\theta_2}} \frac{M(t^{1/2\mu})}{t-\zeta^{2\mu}}\,dt.$$

Now in the sector between L_{θ_1} and L_{θ_2}, the only pole is at $t = \zeta^{2\mu}$, thus closing the contour at infinity and using Cauchy's theorem give

$$f_2(\zeta) = f_2(\zeta e^{-2i\alpha}) + M(\zeta).$$

Continuation to $-4\alpha - \frac{1}{2}\pi < \arg\zeta < \alpha + \frac{1}{2}\pi$ *and check on* (A 1)

Choose $0 < \beta < 2\alpha$, then define the function $F(\zeta e^{-2i\alpha})$ by

$$F(\zeta e^{-2i\alpha}) = N_{2\alpha-\beta-\frac{1}{2}\pi}(\zeta) - M(\zeta) \quad \text{for} \quad -\beta - \tfrac{1}{2}\pi < \arg\zeta < 2\alpha - \beta - \tfrac{1}{2}\pi;$$

in this region both N and M have no poles. The function $F(\zeta e^{-2i\alpha})$ is an analytic function, and for $-\frac{1}{2}\pi < \arg\zeta < 2\alpha - \beta - \frac{1}{2}\pi$

$$F(\zeta e^{-2i\alpha}) = f_2(\zeta) - M(\zeta) = f_2(\zeta e^{-2i\alpha}).$$

Thus $F(\zeta)$ is an analytic continuation of $f_2(\zeta)$ into $-4\alpha - \frac{1}{2}\pi < \arg\zeta \leqslant -2\alpha - \frac{1}{2}\pi$ and from its definition $f(\zeta e^{-2i\alpha}) = f(\zeta) - M(\zeta)$ for $-2\alpha - \frac{1}{2}\pi < \arg\zeta$. Hence we have an analytic continuation of f_2 into $-4\alpha-\frac{1}{2}\pi < \arg\zeta < \alpha+\frac{1}{2}\pi$ which satisfies (A 1) in $-2\alpha-\frac{1}{2}\pi < \arg\zeta < \alpha+\frac{1}{2}\pi$, and therefore in $-\pi-\frac{1}{2}\alpha \leqslant \arg\zeta \leqslant \pi-\frac{1}{2}\alpha$. Note that, in the evaluation of $f_2(e^{\frac{1}{2}i\pi})$, either the contour L_θ must be chosen such that $\theta > \frac{1}{2}\pi$, or if $\theta = 0$, the relation (A 1) must be used.

The expansion of $f_2(\zeta)$ for small ζ

The integral representation

$$f_2(\zeta) = \frac{-\mu}{\pi i}\int_0^\infty \frac{s^{2\mu-1}M(s)}{s^{2\mu}-\zeta^{2\mu}}\,ds$$

valid for $-2\alpha < \arg\zeta \lesssim 0$ is expanded for small ζ by the following device. Choose γ such that $|\zeta| \ll \gamma \ll 1$ and write

$$f_2(\zeta) = -2\mu^{\frac{1}{2}}e^{i\alpha}\sin\alpha(I_1+I_2)/a\pi,$$

where

$$I_1 = \int_0^\gamma \frac{s^{2\mu}(s+i)\dfrac{d^2}{ds^2}\left[\dfrac{s-i}{s+i}f_1(s)\right]}{(s^{2\mu}-\zeta^{2\mu})(s-i)f_1(s)}\,ds, \quad I_2 = \int_\gamma^\infty \ldots ds.$$

In the first integral I_1, s takes only values $0 \leqslant s \leqslant \gamma \ll 1$, thus the function $f_1(s)$ is replaced by its expansion for small s (found by the same method). I_1 becomes

$$\begin{aligned} I_1 &= \mu\int_0^\gamma \frac{(\mu-1)s^{2\mu-2}-2(\cot\alpha-i)s^{2\mu-1}+O(s^{4\mu-2})}{s^{2\mu}-\zeta^{2\mu}}\,ds \\ &= \left(\frac{(\mu-1)\pi}{2\zeta}+2\mu\log\zeta+i\pi\right)\frac{e^{-i\alpha}}{\sin\alpha}-\frac{\mu(\mu-1)}{\gamma}-2\frac{e^{-i\alpha}\mu\log\gamma}{\sin\alpha}+o(1). \end{aligned}$$

In the integral I_2, s always takes values $s \gg |\zeta|$, so the factor $(s^{2\mu}-\zeta^{2\mu})^{-1}$ is expanded for small ζ. We need only the first term, i.e.

$$I_2 = \int_\gamma^\infty \frac{F''}{F}\,ds + o(1), \quad F = \frac{s-i}{s+i}f_1(s).$$

This is too difficult to evaluate explicitly, however

$$F''/F = (\mu-1-2s(\cot\alpha-i))/s^2+O(1) \quad \text{as} \quad s\to 0.$$

Now

$$\int_\gamma^\infty E\,ds \equiv \int_\gamma^\infty \left(\frac{\mu(\mu-1)}{s^2}-\frac{2i\mu(\cot\alpha-i)}{s(s+i)}\right)ds = \frac{\mu(\mu-1)}{\gamma}+2\mu(\cot\alpha-i)\left(\log\gamma-\frac{i\pi}{2}\right).$$

Therefore

$$f_2(\zeta) = \frac{\mu^{\frac{1}{2}}}{a\pi}\left\{-\frac{\pi(\mu-1)}{\zeta}-4\mu\log\zeta-2\left[e^{i\alpha}\sin\alpha\int_0^\infty\left(\frac{F''}{F}-E\right)ds+i\pi(1-\mu)\right]\right\}+o(1).$$

Hence the limit of $f_1(\zeta)f_2(\zeta)/\zeta$ as $\zeta\to 0$ is obtained; thus for large Z

$$\begin{aligned} W_1(Z) \sim \frac{\Gamma(\mu)\mu^{\frac{1}{2}}}{a\pi^2}\{&i\pi Z^{1-\mu}\sin\pi\mu+4i\mu Z^{-\mu}\log Z\sin\pi\mu+K_1\sin\pi\mu Z^{-\mu} \\ &-i[K_2\sin\pi\mu+4\mu\pi\cos\pi\mu]Z^{-\mu}+\ldots\}, \end{aligned}$$

where

$$K_2+iK_1 = \frac{2}{\pi}e^{i\alpha}\sin\alpha\int_0^\infty\left(\frac{F''}{F}-E\right)ds+2i(1-\mu)+(\mu-1)\frac{e^{i\alpha}}{\sin\alpha}+4\mu\psi(\mu),$$

K_2 and K_1 are real and $\psi(\mu)$ is the digamma function $\Gamma'(\mu)/\Gamma(\mu)$.

The expansion of $f_3(\zeta)$ for small ζ

The function $f_3(\zeta)$ is defined for $-2\alpha < \arg\zeta < 0$ by the Plemelj integral

$$f_3(\zeta) = -2\frac{\mu^{\frac{3}{2}}e^{i\alpha}\sin\alpha}{\alpha\pi}\int_0^\infty\frac{s^{4\mu}(s+i)\dfrac{d^2}{ds^2}\left[\dfrac{s-i}{s+i}s^{-2\mu}f_1(s)\right]}{(s^{2\mu}-\zeta^{2\mu})(s-i)f_1(s)}ds.$$

The expansion of this function as $\zeta\to 0$ is almost exactly the same as the expansion of $f_2(\zeta)$. Recall that $\Omega(Z_1) = L(\zeta^{-2\mu}f_1f_3)$; we find

$$\Omega(Z_1)\sim\frac{-i\mu^{\frac{1}{2}}}{\alpha\pi\Gamma(\mu)}[\pi Z_2^{1+\mu}+4\mu Z_1^{\mu}\log Z_1+LZ_1^{\mu}+\ldots],$$

where

$$L = L_2+iL_1 = 2e^{i\alpha}\sin\alpha\int_0^\infty\left\{\frac{\left[\dfrac{s-i}{s+i}s^{-2\mu}f_1\right]''}{\dfrac{s-i}{s+i}s^{-2\mu}f_1}-\frac{\mu(\mu+1)}{s^2}-\frac{2i\mu(\cot\alpha-i)}{s(s+i)}\right\}ds$$
$$-2i\pi(1-\mu)-4\mu\psi(1+\mu)-(\mu+1)(\cot\alpha+i)\pi.$$

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